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The Existence of Bounded Solutions of a Semilinear Elliptic Equation

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1. INTRODUCTION

In this paper, we shall study the existence of the bounded nonconstant solution of the differential equation

$$\Delta w - y \cdot \nabla w/2 + |w|^{p-1} w - w/(p-1) = 0 \tag{1.1}$$

in $\mathbb{R}^n$, $n > 2$ with $p > p_c$, where $p_c \equiv (n+2)/(n-2)$ is the critical Sobolev space exponent for $\mathbb{R}^n$, $y \cdot \nabla w = \sum_{j=1}^{n} y_j(\partial w/\partial y_j)$. Equation (1.1) is derived from the heat equation

$$u_t - \Delta u - |u|^{p-1} u = 0 \tag{1.2}$$

which, because of the superlinearity of the nonlinear term, has the property that an initially bounded solution may become infinite in finite time, i.e., its solutions may blow up in finite time. The reader is refered to [2, 4, 5] and the references given there for the study of blow-up of solutions to (1.2) and related topics. In particular, Giga and Kohn made the key observation in [4] and [5] that, if we assume a solution u of (1.2) blows up at the point $x = 0$ when $t \to T > 0$ and let

$$w(y) = \operatorname*{Lim}_{t \to T} (T-t)^{1/(p-1)} u(x, t),$$

where $y = x/(T-t)^{1/2}$, then w, the "backward self-similar solution" of the heat equation, satisfies (1.1). Thus, the study of differential equation (1.1) is very important in our understanding of the exact behaviour of solutions of (1.2) near blow-up time. Indeed, Giga and Kohn proved in [4] that (1.1) has no nontrivial globally bounded solutions for $n = 1, 2$, or $n > 2$, $p \leqslant p_c$ except the two constant solutions

$$w \equiv \pm \beta^{\beta}, \tag{1.3}$$

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where $\beta = 1/(p-1)$. Using this fact they gave a characterization of asymptotic behaviours of solutions of (1.2) near the blow-up time. More recently, Bebernes and Eberly [1] showed that for radially symmetric solutions of (1.2) on a ball with initial data satisfying $u_0 \geqslant 0$, $\Delta u_0 + u_0^p \geqslant 0$, the blow-up is asymptotically like the constant solution of (1.1) even for $p > p_c$. However, Giga [3] has recently shown that (1.1) does have a radially symmetric bounded solution if the term $w/(p-1)$ is replaced by αw, $\alpha > 1/(p-1)$, $n > 2$ and $p \leqslant p_c$. In addition, he showed that if $\alpha \leqslant 1/(p-1)$ and $p < p_c$, then there are no radially symmetric solutions.

In this paper, we shall study, in contrast, Problem (1.1) with parameter range $p > p_c$ and consider the radially symmetric solutions w which are functions of the variable r only where $r = |y|$, so (1.1) becomes in this case

$$w'' + \left(\frac{n-1}{r} - \frac{r}{2}\right) w' + |w|^{p-1} w - w/(p-1) = 0. \tag{1.4}$$

A bounded solution of (1.4) must satisfy the initial conditions

$$w(0) = \alpha \in \mathbb{R}, \qquad w'(0) = 0. \tag{1.5}$$

The standard theory of initial value problems implies the existence of such a solution in a neighbourhood of the origin. However, we assume throughout that on each compact interval $[0, L] \subset [0, +\infty)$ the solution of (1.4), (1.5) exists, is unique, and depends continuously on the initial data (cf. [8]). We now state the main result of this paper.

THEOREM 1. *For* $2 < n \leqslant 10$ *and* $p > p_c$ *or for* $n \geqslant 11$ *and* $p_c < p < [n - 2(n-1)^{1/2}]/[n - 4 - 2(n-1)^{1/2}]$ *there is an unbounded increasing positive sequence* $\{\alpha_L\}$, $L \in \mathbb{N}$ *such that for each L, if w is the solution of* (1.4) *and* $w(0) = \alpha_L$, $w'(0) = 0$ *then* $w(r) > 0$ *for all* $r > 0$ *and* $\lim_{r \to \infty} w(r) = 0$. *Furthermore,* $0 < w(r) < [(2(n-2)\, p - 2n)/(p-1)^2\, r^2]]^{1/(p-1)}$ *for all large r.*

This result extends the work of Troy [12] who proved the above result for $n = 3$ and $6 \leqslant p \leqslant 12$. Also, the authors of [6] gave an explicit, smooth, decaying solution of (1.1) for $p = 2$ and $6 < n < 16$. We note, however, that $p = 2$ for $6 < n < 16$ is still in the range we gave in Theorem 1. In this paper we use similar methods to those of [12] but employ some new techniques which allow us to extend [12] to the range given in Theorem 1. We also show how the proof may be interpreted geometrically by using a dynamical system and give an explanation of how the odd restriction on the exponent appears.

In Section 2 we give a proof of Theorem 1. In Section 3 we study the global behaviour of the solutions of (1.4), (1.5) and, in particular, we shall establish the following result.

THEOREM 2. *Let $n > 2$ and $p > p_c$; then there exists no solution of* (1.4), (1.5) *such that*

$$\lim_{r \to \infty} w(r) = \pm \beta^\beta \qquad \text{and} \qquad w(r) \not\equiv \pm \beta^\beta.$$

2. THE EXISTENCE RESULT

In this section we consider only the exponent range given in Theorem 1 unless otherwise stated. It is easy to verify that for $n \geqslant 2$ and $p > p_c$ the problem (1.4) has the singular solution

$$w_0(r) = [(2(n-2)(p-1)-4)/(p-1)^2 r^2]^{1/(p-1)}. \tag{2.1}$$

It follows by substitution that $w_0(r)$ is also a solution of the ordinary differential equation

$$v'' + \frac{n-1}{r} v' + v^p = 0. \tag{2.2}$$

It is well known (cf. [10]) that the solution v of the differential equation (2.2) which has the initial value

$$v(0) = 1, \qquad v'(0) = 0 \tag{2.3}$$

is positive and bounded for all $r > 0$, $\lim_{r \to \infty} v(r) = 0$, and $v(r)$ intersects the function $w_0(r)$ an infinite number of times (for the exponents given in Theorem 1). This leads us to define for solutions $w = w(r, \alpha)$ of (1.4), (1.5) described by Troy [12] the set

$$A_{2L} = \{\alpha > \beta^\beta \mid w - w_0 \text{ has at least } 2L + 2 \text{ zeros before } w = 0\}.$$

From the continuous dependence of solutions on the initial value one can easily deduce that A_{2L} is open.

Let $u(r) = w(\alpha^{-(p-1)/2} r)/\alpha$, where w is a solution of (1.4), (1.5); then u satisfies the ordinary differential equation

$$\begin{gathered} u'' + \frac{n-1}{r} u' + |u|^{p-1} u - \alpha^{1-p} \left(\frac{r}{2} u' + \frac{u}{p-1} \right) = 0, \\ u(0) = 1, \qquad u'(0) = 0. \end{gathered} \tag{2.4}$$

We proceed to prove that A_{2L} is open, nonempty, and unbounded for every positive integer L. First, we have the following

LEMMA 1. *For every fixed $\lambda > 0$ there exists $\alpha(\lambda) > 0$ such that for $\alpha > \alpha(\lambda)$ the boundary value problem*

$$\Delta u + u^p - \alpha^{1-p}\left(\frac{x \cdot \nabla u}{2} + \frac{u}{p-1}\right) = 0, \qquad u|_{\partial B} = 0, \qquad u \geqslant 0 \text{ in } B \tag{2.5}$$

has no nontrivial solution on $B = \{x \in \mathbb{R}^n \mid |x| \leqslant \lambda\}$.

COROLLARY. *For every fixed $\lambda > 0$ the solution of* (2.4) *is positive on* $[0, \lambda]$, *if α is sufficiently large.*

Proof of Lemma 1. Suppose to the contrary and without loss of generality we assume

$$\alpha > \left(\frac{\lambda^2(p+2)}{(n-2)(p+1) - 2n}\right)^{1/(p-1)}.$$

We now define the two functions

$$F(x, u, q) \equiv |q|^2/2 + \frac{\alpha^{1-p}}{2(p-1)} u^2 - |u|^{p+1}/(p+1), \qquad g(x, u, q) \equiv \frac{x \cdot q}{2} \alpha^{1-p}; \tag{2.6}$$

here $q = (q_1, ..., q_n)$, then the differential equation in (2.5) is equivalent to

$$\operatorname{div} F_q(x, u, \nabla u) = F_u(x, u, \nabla u) + g(x, u, \nabla u). \tag{2.7}$$

We shall now employ the following general Pohožaev identity which is derived by Pucci and Serrin [11] and which we state precisely in the following lemma:

LEMMA 2. *Let u be a C^2 solution of* (2.6) *defined in a bounded domain $\Omega \in \mathbb{R}^n$ with $u = 0$ on the boundary $\partial\Omega$ of Ω. Then*

$$\oint_{\partial\Omega} \left[F(x, 0, \nabla u) - \frac{\partial u}{\partial x_i} F_{q_i}(x, 0, \nabla u)\right](h \cdot \nu)\, ds$$

$$= \int_\Omega \Big\{F(x, u, \nabla u) \operatorname{div} h + h_i F_{x_i}(x, u, \nabla u)$$

$$- \left[\frac{\partial u}{\partial x_i}\frac{\partial h_i}{\partial x_j} + u \frac{\partial a}{\partial x_j}\right] F_{q_j}(x, u, \nabla u)$$

$$-a\left[\frac{\partial u}{\partial x_i}F_{q_i}(x,u,\nabla u)+uF_u(x,u,\nabla u)\right]$$

$$-\left[h_j\frac{\partial u}{\partial x_j}+au\right]g(x,u,\nabla u)\Big\}\,dx;$$

here α and h are functions lying in $C^1(\Omega)\cap C(\bar{\Omega})$ and repeated indices i and j are understood to be summed from 1 *to n.*

For the proof of this result, see [11].

Applying the above general Pohožaev Identity with $\alpha=\text{const}$ and $h=x$ on B to our problem (written in the form of (2.7)) yields

$$-\frac{\lambda}{2}\oint_{\partial B}|\nabla u|^2\,ds=\int_B\left[\left(\frac{n-2}{2}-a\right)|\nabla u|^2-\alpha^{1-p}|x\cdot\nabla u|^2/2-aux\cdot\nabla u\right.$$
$$\left.+\left(\frac{n}{2}-a\right)\alpha^{1-p}u^2/(p-1)+\left(a-\frac{n}{p+1}\right)u^{p+1}\right];$$

if we take $a=n/(p+1)$ then we have

$$-\frac{\lambda}{2}\oint_{\partial B}|\nabla u|^2\,ds$$
$$=\int_B\left\{\left(\frac{n-2}{2}-\frac{n}{p+1}\right)|\nabla u|^2+\left(\frac{n}{2}-\frac{n}{p+1}\right)\alpha^{1-p}u^2/(p-1)\right.$$
$$\left.-\alpha^{1-p}|x\cdot\nabla u|^2/2-n\alpha^{1-p}ux\cdot\nabla u/(p+1)\right\}dx$$
$$\geqslant\int_B\left\{\left(\frac{n-2}{2}-\frac{n}{p+1}-\alpha^{1-p}|x|^2/2\right)|\nabla u|^2\right.$$
$$\left.+\left(\frac{n}{2}-\frac{n}{p+1}\right)\alpha^{1-p}u^2/(p-1)-n\alpha^{1-p}ux\cdot\nabla u/(p+1)\right\}dx.$$

By our assumption on α we have, for the right hand side terms of the above inequality

$$\frac{n-2}{2}-\frac{n}{p+1}-\alpha^{1-p}|x|^2/2$$
$$>\frac{n-2}{2}-\frac{n}{p+1}-\frac{(n-2)(p+1)-2n}{2\lambda^2(p+2)}|x|^2$$
$$\geqslant\frac{(n-2)(p+1)-2n}{2(p+1)}-\frac{(n-2)(p+1)-2n}{2(p+2)}>0$$

and

$$-\int_B ux\cdot\nabla u\,dx = -\int_B [\operatorname{div}(xu^2/2) - nu^2/2]\,dx$$
$$= \frac{n}{2}\int_B u^2\,dx - \oint_{\partial B} xu^2/2\,ds$$
$$= \frac{n}{2}\int_B u^2\,dx \geqslant 0.$$

Here the last identity follows from the boundary condition. So the right hand side of the above formula is positive since $p > 1$, but clearly the left hand side is nonpositive and we reach a contradiction. Therefore we have completed the proof of Lemma 1. ∎

Next we shall prove that the solutions of (2.4) and their first derivatives are uniformly bounded in the supremum norm on any fixed interval $[0, \lambda]$ for all large α. In fact, we have the following

LEMMA 3. *For any $\lambda > 0$, the solutions $u(r, \alpha)$ of (2.4), when $\alpha > \alpha(\lambda)$, are uniformly bounded in $C^1([0, \lambda])$. That is, there exists a function $k(\lambda)$ independent of α such that for all $\alpha > \alpha(\lambda)$*

$$|u(r, \alpha)| + |u'(r, \alpha)| < k(\lambda).$$

Proof. From Lemma 1 we know that $u(r, \alpha)$ is positive on the interval $[0, \lambda]$. If we multiply the differential equation in (2.4) by r^{n-1} and integrate over $[0, r]$ we have

$$\begin{aligned} u'r^{n-1} &= -\int_0^r s^{n-1}u^p\,ds + \alpha^{1-p} \\ &\quad \times \int_0^r [s^n u'/2 + s^{n-1}u/(p-1)]\,ds \\ &= -\int_0^r s^{n-1}u^p\,ds + \alpha^{1-p} s^n u/2|_0^r \\ &\quad - \int_0^r \left[\left(\frac{n}{2} - \frac{1}{p-1}\right) s^{n-1}u\right] ds. \end{aligned} \tag{2.8}$$

This yields

$$u'r^{n-1} \leqslant \alpha^{1-p} r^n u/2. \tag{$*$}$$

Integrating the above inequality and by noting that $u(0, \alpha) = 1$ we have

$$0 < u \leqslant \exp(\alpha^{1-p} r^2/4). \tag{2.9}$$

If we substitute (2.7) into (*) and (2.8) we see that u' is bounded from both above and below and there exists a function $k(\lambda)$ depending only on λ such that $|u'| \leqslant k(\lambda)$. This completes the proof of Lemma 3. ▮

We observe that to prove A_{2L} is nonempty and unbounded it is sufficient to prove that for α sufficiently large $|u(r, \alpha) - v(r)|$ tends to zero uniformly for $r \in [0, \lambda]$ as α tends to infinity because v intersects w_0 an infinite number of times. Here as before, λ is a fixed positive number and v is the solution of (2.3). For this end, we have

LEMMA 4. *Let $\lambda > 0$ and suppose that α is sufficiently large; then the following inequality holds on $[0, \lambda]$;*

$$|u(r, \alpha) - v(r)| \leqslant M\alpha^{1-p} e^{Mr^2} \int_0^r s^2 e^{-Ms^2}\, ds.$$

Where $M = M(\lambda)$ is a function depending only on λ.

Proof. Suppose $\alpha > \alpha(\lambda)$; then we may recast the ordinary differential problems (2.2), (2.3), and (2.4) into the following integral forms which may be deduced by integrating the corresponding differential equations:

$$v = 1 - \int_0^r s\left(1 - \frac{s}{r}\right) v^p\, ds,$$

$$u(r, \alpha) = 1 - \int_0^r s\left(1 - \frac{s}{r}\right) u^p\, ds + \alpha^{1-p} \int_0^r s\left(1 - \frac{s}{r}\right)\left(\frac{su'}{2} + \frac{u}{p-1}\right) ds.$$

Subtracting these two expressions and using the bound derived in Lemma 3 we may deduce

$$|u - v| \leqslant M\left[\int_0^r s|u - v|\, ds + \alpha^{1-p} r^2\right].$$

In an application of Gronwall's Lemma, Hartman [7] then gives

$$|u - v| \leqslant M\alpha^{1-p} e^{Mr^2} \int_0^r s^2 e^{-Ms^2}\, ds. \quad ▮$$

COROLLARY. *A_{2L} is open, nonempty, and unbounded for every positive integer L.*

Next we state a result which was first proved by Troy [12] and shows another property of the solutions of Problems (1.4), (1.5).

LEMMA 5. *If $\alpha - \beta^{\beta} > 0$ is sufficiently small then $h \equiv w - w_0$ has at most two zeros before $w = 0$.*

Proof. See Troy [12].

Proof of Theorem 1. We observe that from the above Corollary we know that for any positive integer L, A_{2L} is open, nonempty, and unbounded. So, if we combine this result with Lemma 5, we may now employ the argument of [12] to establish the desired results, but for simplicity we shall omit it here. ∎

We now show, for completeness, how the steps in our proof may be interpreted geometrically by formulating our problem as a dynamical system. We make the change of variables (described in Jones and Küpper [9])

$$a(t) = r^{\sigma} u,$$
$$b(t) = r^{\sigma+1} u'$$

and

$$t = \log(r),$$

where $\sigma = 2/(p-1)$. The problem (1.4) is then transformed into the dynamical system

$$\begin{aligned} \frac{da}{dt} &= \sigma a + b, \\ \frac{db}{dt} &= (\sigma + 2 - n) + (\sigma a + b)\, r^2/2 - a|a|^{p-1}, \\ \frac{dr}{dt} &= r. \end{aligned} \tag{2.10}$$

The solution v of (2.2), (2.3) satisfies, correspondingly,

$$\begin{aligned} \frac{da}{dt} &= \sigma a + b, \\ \frac{db}{dt} &= (\sigma + 2 - n) - a|a|^{p-1}. \end{aligned} \tag{2.11}$$

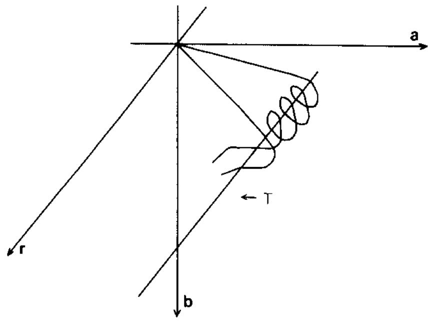

FIGURE 1

The singular solution $w_0(r)$ is precisely the equilibrium point $S=(a_0, b_0)$, where $a_0=[\sigma(n-2-\sigma)]^{1/(p-1)}$ and $b_0=-\sigma a_0$. It is easy to prove that for $3 \leqslant n \leqslant 10$, S is a stable spiral for all $p > p_c$. But when $n \geqslant 11$, S is a stable spiral only for exponents $p_c < p < [n-2(n-1)^{1/2}]/[n-4-2(n-1)^{1/2}]$, whereas for (2.10), a solution of (1.4, 5) corresponds to a solution trajectory of (2.10) which is asymptotic to $(0, 0, 0)$ as $t \to -\infty$. Further, the singular solution $w_0(r)$ is precisely the trajectory $T=(a_0, b_0, r)$, where $a_0=[\sigma(n-2-\sigma)]^{1/(p-1)}$ and $b_0=-\sigma a_0$. This curve is a straight line in the phase space $\mathbb{R}^3$ of (a, b, r). From Lemma 5 we may deduce that if the initial value α is close to β^β the corresponding solution trajectory of (2.10) leaves the origin and loops once around T before intersecting the plane $a=0$ at which point $u(r)=0$. Moreover, if α is sufficiently large the implications of the Corollary tell us that the corresponding solution trajectory may loop an arbitrary number of times around T. This behaviour is indicated in Fig. 1. The argument given in [12] says that we can obtain the desired solutions stated in Theorem 1 as we continuously deform the trajectory given when α is small into that given when α is large.

3. The Asymptotic Behaviour of Bounded Solutions

In this section, we prove the following result:

THEOREM 3. *Every bounded solution u of* (1.4), (1.5) *will tend to zero and for each bounded positive solution there exists a constant* $C>0$ *such that* $u \geqslant Cr^{-2/(p-1)}$ *for all r large.*

Remark. Theorem 2 is an immediate consequence of the above result.

The first step of our proof is to show that any bounded solution of (1.4), (1.5) will tend to zero as $r \to \infty$. However, we first need the following two technical lemmas.

LEMMA 6. *Let u be a bounded solution of* (1.4), (1.5); *then the first derivative $u'(r)$ will tend to zero as $r \to \infty$.*

Proof. If we multiply Eq. (1.4) by $e^{-r^2/4}r^{n-1}$ and integrate over $[r, R]$, $R > r > 0$, then we have

$$u'(t)\, t^{n-1} e^{-t^2/4} \big|_r^R = \int_r^R s^{n-1} e^{-s^2/4} \left(-|u|^{p-1} u + \frac{u}{p-1} \right) ds. \tag{3.1}$$

As u is bounded we know that there exists a sequence $\{R_m\}_1^\infty$ of numbers such that $|u'(R_m)| \leqslant 1$ and $\lim_{m \to \infty} R_m = \infty$. Take $R = R_m$ on the left hand side of (3.1) and take the limit; then we have

$$-u'(r)\, r^{n-1} e^{-r^2/4} = \int_r^\infty s^{n-1} e^{-s^2/4} \left(-|u|^{p-1} u + \frac{u}{p-1} \right) ds.$$

This implies that

$$-u'(r) = \frac{\displaystyle\int_r^\infty s^{n-1} e^{-s^2/4} \left(-|u|^{p-1} u + \frac{u}{p-1} \right) ds}{r^{n-1} e^{-r^2/4}}.$$

Furthermore, we may deduce from L'Hôpital's rule that

$$\lim_{r \to \infty} \frac{\int_r^\infty s^{n-1} e^{-s^2/4}\, ds}{r^{n-1} e^{-r^2/4}} = 0.$$

Thus, combining this result with the boundedness of $u(r)$, we obtain the desired result and the proof is complete. ∎

LEMMA 7. *If u is a solution of* (1.4), (1.5) *which is positive, bounded in* $[0, +\infty]$, *and does not tend to zero as $r \to \infty$, then there exist an infinite number of zeros of function $u - \beta^\beta$.*

Proof. Let us suppose the contrary; then $u - \beta^\beta$ will be either strictly positiver or negative for all large r. Let us assume that $u - \beta^\beta < 0$ for all $r < r_0 \equiv \sqrt{2(n-1)}$. If for some $r_1 > r_0$ we have $u'(r_1) \geqslant 0$ then from Eq. (1.4) we obtain $u'' > 0$ provided that $u < \beta^\beta$; then there must exist a point at which $u = \beta^\beta$ and we reach a contradiction. This means u must be nonincreasing and so, as u does not tend to zero, u tends to some positive constant C, $0 < C < \beta^\beta$. From the proof of Lemma 6 we see that $u'(r)r$ will tend to some negative constant and it follows that u will tend to minus infinity; this is clearly a contradiction. Thus we complete the proof of Lemma 7 for this case. A similar argument applies for the case $u - \beta^\beta > 0$. ∎

Proof of Theorem 3. Firstly, we assume that u is a positive bounded solution of (1.4), (1.5). Suppose to the contrary that u does not tend to zero as $r \to \infty$. Let $E(r)$ denote the function

$$E(r) \equiv \frac{(u'(r))^2}{2} - \frac{u^2}{2(p-1)} + \frac{u^{p+1}}{p+1}.$$

From Eq. (1.4) we know that

$$E'(r) = \left(\frac{r}{2} - \frac{n-1}{r}\right)(u'(r))^2 \geqslant 0, \qquad r \geqslant r_0,$$

and from Lemma 6 we have $\lim_{r \to \infty} u'(r) = 0$. On the other hand, Lemma 7 implies that there exists a sequence $\{x_n\}_1^\infty$ such that $\lim_{r \to \infty} x_n = \infty$ and $u - \beta^\beta = 0$ at each x_n. As $E(r)$ is nondecreasing this yields

$$|u'(x_{n+1})| > |u'(x_n)| > 0, \qquad n \geqslant 1.$$

This is impossible, and this contradiction enables us to establish that $\lim_{r \to \infty} u(r) = 0$. For a general bounded solution u of (1.4), we know from the above proof that it cannot have an infinite number of points where u is equal to zero because $E(r)$ is non-decreasing function of r and $u'(r)$ tends to zero as $r \to \infty$. So $u(r)$ will be either strictly positive or negative for all r large. The above argument applies to the positive case and the negative case is similar. This establishes that for any bounded solution u of (1.4), (1.5), $\lim_{r \to \infty} u(r) = 0$.

We know from the above that u and u' tend to zero as $r \to \infty$, and from the proof of Lemma 7 we see that if u is positive for all r large, then u' will be negative for all r large, so u'' will not be negative for all r large. In fact, differentiating Eq. (1.4) gives

$$u''' = \left(\frac{r}{2} - \frac{n-1}{r}\right)u'' + \left(\frac{1}{2} + \frac{n-1}{r^2}\right)u' + \left(\frac{1}{p-1} - pu^{p-1}\right)u',$$

and from this it is easy to deduce that $u'' > 0$ for all r large. This implies, from Eq. (1.4), that

$$\left(\frac{r}{2} - \frac{n-1}{r}\right)u' \geqslant -\frac{u}{p-1};$$

integrating this inequality then yields

$$\log(u)|_{r_1}^{r} \geqslant -\frac{1}{p-1}\log(r^2 - 2(n-1)), \qquad r, r_1 \text{ large},$$

and so

$$u \geqslant Cr^{-2/(p-1)}, \qquad C = \text{const} > 0 \text{ and } r \text{ large.}$$

This completes the proof of Theorem 3. ∎

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